Structured Clique Networks as Efficient Associative Memories

Vincent Gripon

IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

BRAIn

Lab-STICC

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Outline

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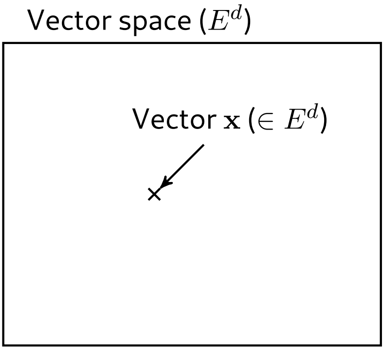
1. Context
2. Associative Memories
   - Hopfield Neural Networks
   - Willshaw Neural Networks
3. Structured Clique Networks
   - Principles
   - Theoretical results
   - Experiments
4. Example Applications
   - Applications in Hardware
   - Approximate Nearest Neighbor Search
   - DecisiveNets
5. Conclusion

Notations and problems

Vector space (E^d)

1. Supervised learning,
2. Unsupervised learning,
3. Indexing,
4. Search…

Notations and problems



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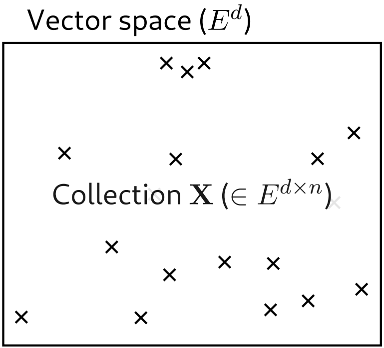

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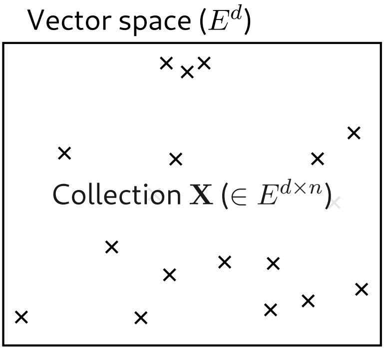

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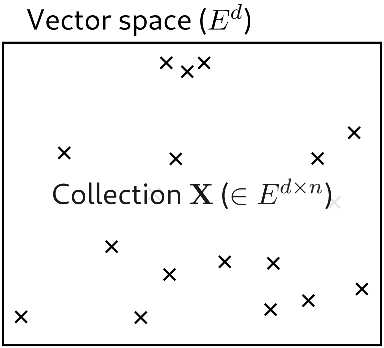

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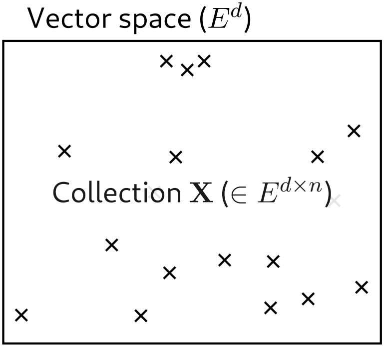

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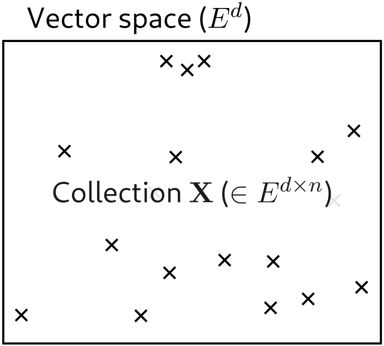

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To learn is to **generalize** ($\neq$ memorize),

Supervised learning

- Regression,
- Requires an expert,
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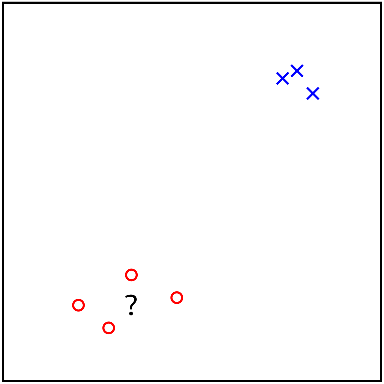

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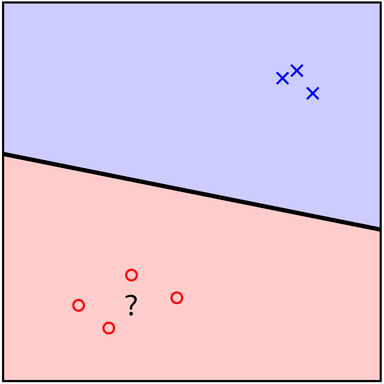

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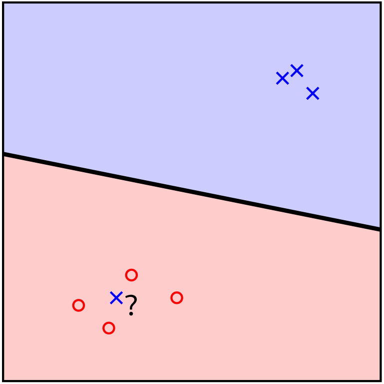

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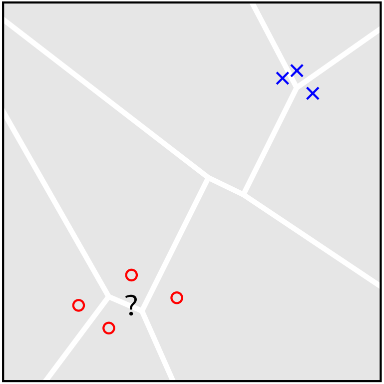

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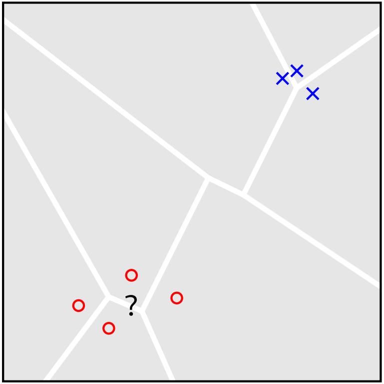

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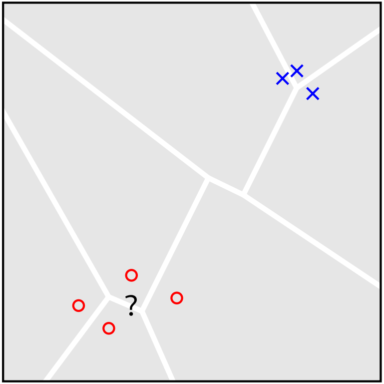

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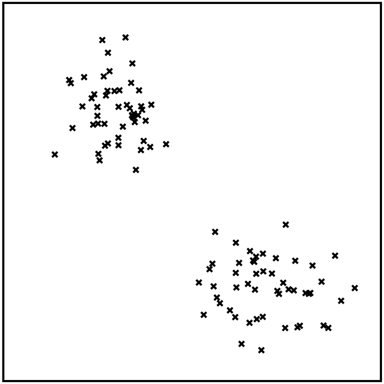

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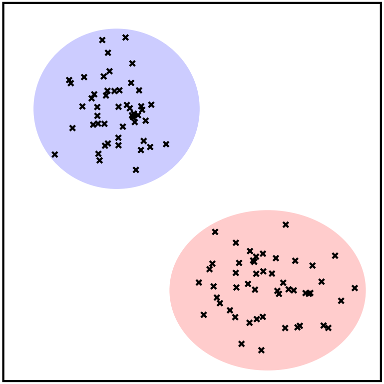

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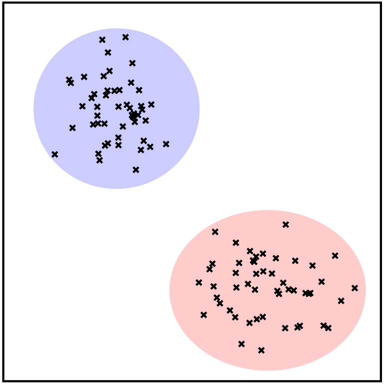

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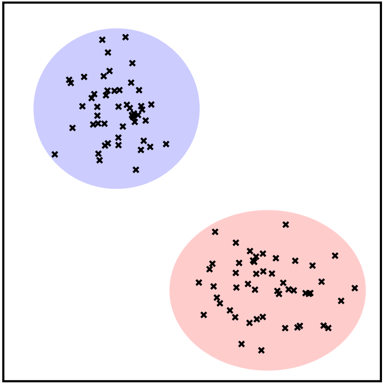

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- Pros: no error, simple, concurrent,
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Example of database: SIFT1B:

- $n = 1,000,000,000$,
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A neural network is a mathematical function obtained by assembling simple ones, called layers, that can be written as:

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Nonlinear functions

- Sigmoids (e.g. $x \mapsto 1/\left(1 + \exp(-x)\right)$),
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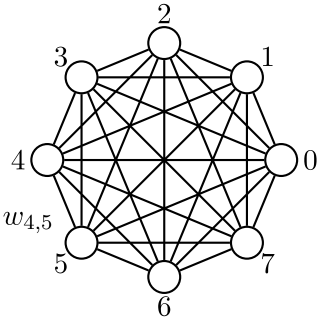

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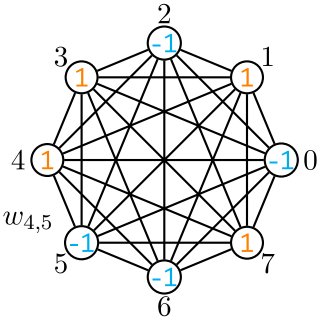

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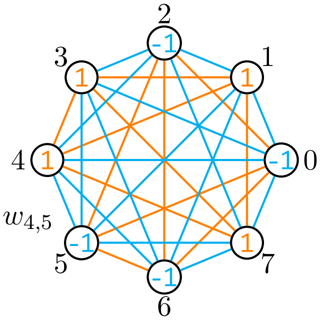

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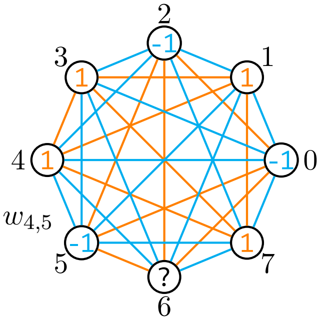

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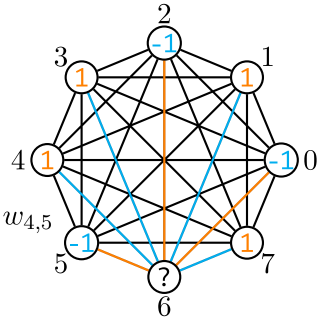

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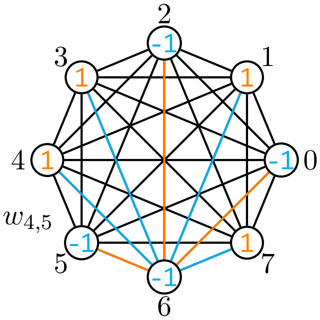

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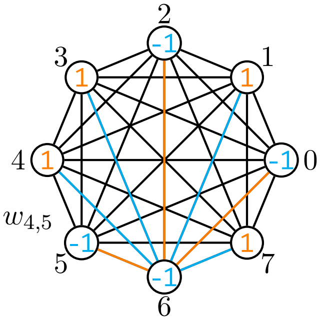

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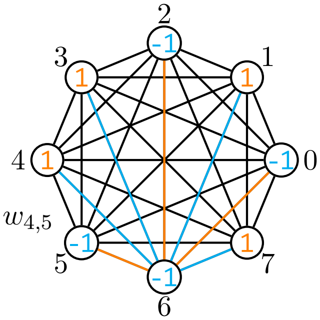

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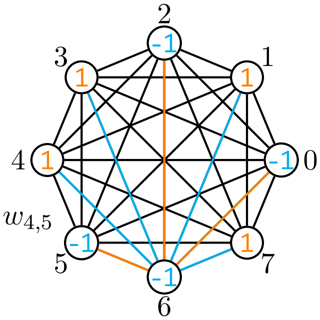

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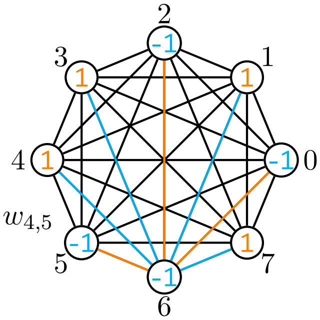

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Theorem [1]

Consider $n = \frac{d}{\gamma \log(d)}$:

- If $\gamma > 6$, then for $d \to \infty$, $\mathbb{P}[\liminf_d \{\cap_{\mathbf{x} \in \mathbf{X}} \{U(\mathbf{x}) = \mathbf{x}\}] = 1$,
- If $\gamma > 4$, then $\mathbb{P}[\cap_{\mathbf{x}} \{U(\mathbf{x}) = \mathbf{x}\}] \to 1$,
- If $\gamma < 2$, then $\mathbb{P}[\cap_{\mathbf{x}} \{U(\mathbf{x}) = \mathbf{x}\}] \to 0$.

Memory efficiency

- $\binom{d}{2}$ connections with $n+1$ possible values each $\Rightarrow$ takes $\binom{d}{2} \log_2(n+1)$ bits without compression,
- To be compared to the entropy of $\mathbf{X} \approx nd$,
- When patterns are stable, we obtain $\eta \leq \frac{1}{\log(d)}$.

[1] "Étude asymptotique d'un réseau neuronal: le modèle de mémoire associative de Hopfield", Franck Vermet

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- If $\gamma > 4$, then $\mathbb{P}[\cap_{\mathbf{x}} \{U(\mathbf{x}) = \mathbf{x}\}] \to 1$,
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Memory efficiency

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[1] "Étude asymptotique d'un réseau neuronal: le modèle de mémoire associative de Hopfield", Franck Vermet

Framework

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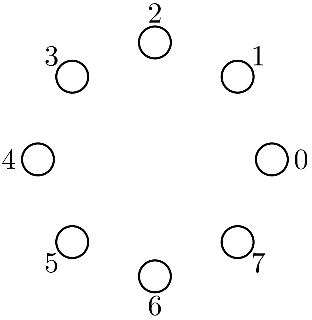

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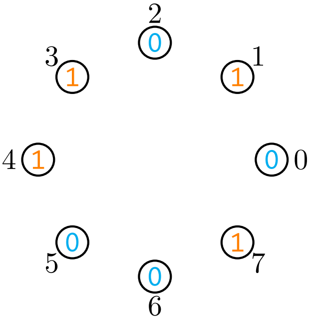

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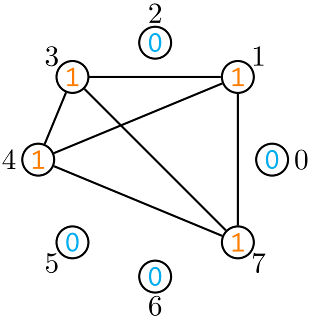

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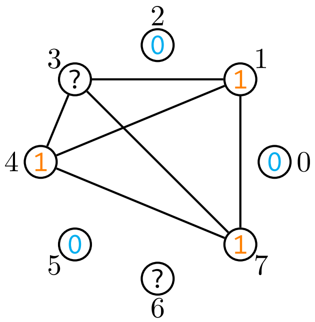

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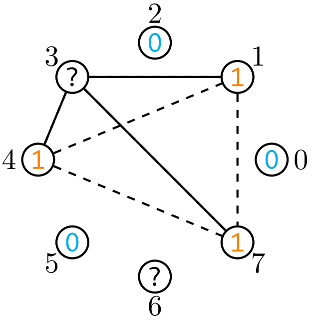

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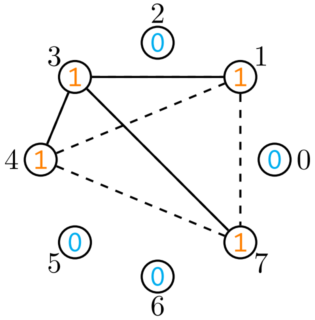

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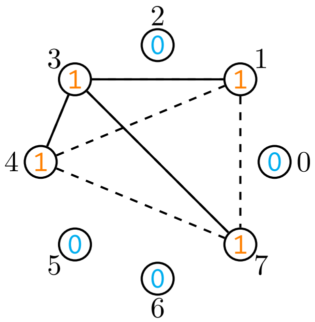

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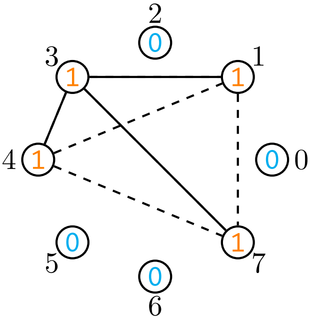

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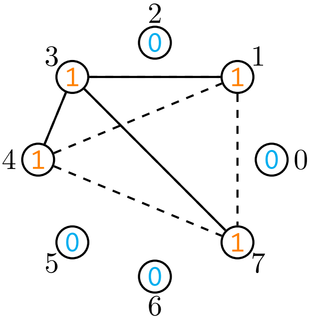

Stability of stored vectors

Theorem [2]

Consider $\mathbf{X}$ generated with $\|\mathbf{x}\|_0 = \lfloor \log(d)/d \rfloor$ and $\mathbf{x}$ chosen at random such that $\|\mathbf{x}\|_0 = \lfloor \log(d)/d \rfloor$. With $n = \alpha d^2 \log\log(d)/\log^2(d)$:

- If $\alpha > 2$, $\mathbb{P}[u(\mathbf{x}) = \mathbf{x}] \to 1$,
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Consider $n = \alpha d^2 / \log^2(d)$, $\rho \in [0,1[$ such that $\lfloor \rho \log(d) \rfloor$ of 1s in $\mathbf{x}$ are erased to obtain $\tilde{\mathbf{x}}$. Then:

- If $\alpha < -\log(1 - \exp(-1/(1-\rho)))$, then $\mathbb{P}[u(\tilde{\mathbf{x}}) = \mathbf{x}] \to 1$,
- If $\alpha > -\log(1 - \exp(-1/(1-\rho)))$, then $\mathbb{P}[u(\tilde{\mathbf{x}}) \neq \mathbf{x}] \to 1$.

On the difficulty of computing the memory efficiency

- $\binom{d}{2}$ connections with 2 possible values each $\Rightarrow$ takes $\binom{d}{2}$ bits without compression,
- To be compared to the entropy of $\mathbf{X}$: $\approx ndH_2(\log(d)/d)$,
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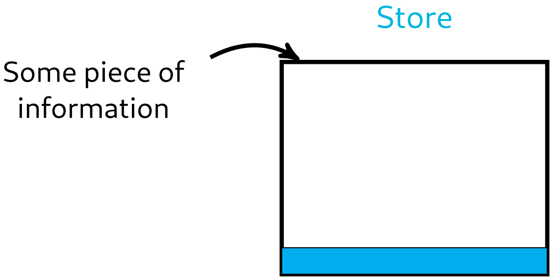
Associative memories

Store

Associative memories



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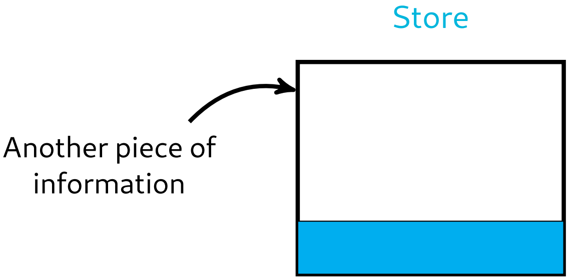

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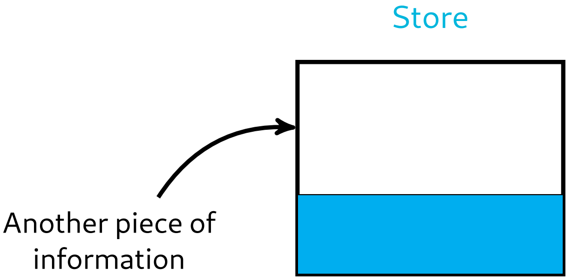

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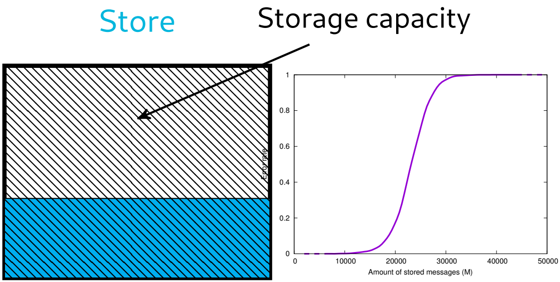

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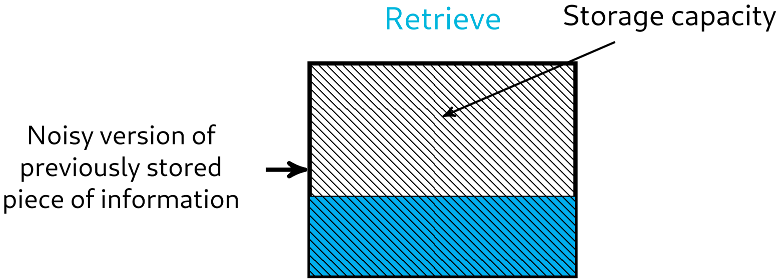
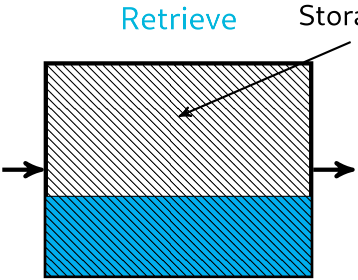


Associative memories

Noisy version of previously stored piece of information



Corresponding previously stored piece of information

Summary

	Hopfield	Willshaw
Framework	$\mathbf{x} \in \{-1,1\}^d$	$\mathbf{x} \in \{0,1\}^d$, $\|\mathbf{x}\|_0 \ll d$
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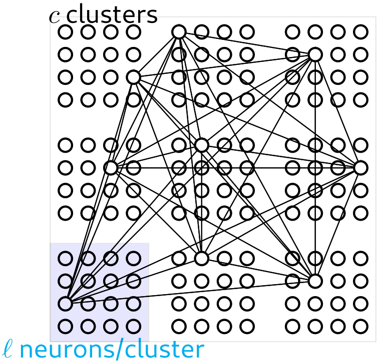
Outline

Adding structure to Willshaw networks



Leveraging the structure

Willshaw rule for retrieval

- $\mathbf{x}_{ij}$ denotes the j-th neuron in the i-th cluster,
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New rule for retrieval

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- $u(\mathbf{x})_{ij} = 1 \Leftrightarrow \sum_{i'} \max_{j'} \mathbf{W}_{i'j',ij}\mathbf{x}_{i'j'}$ is maximal in cluster i,
- $u(\mathbf{x})_{ij} = 1 \Leftrightarrow \wedge_{i'} \vee_{j'} \mathbf{W}_{i'j',ij} \wedge \mathbf{x}_{i'j'}$,
  if neurons in erased clusters are all initialized active.
- Both storage and retrieval are performed using Boolean algebra.

Leveraging the structure

Willshaw rule for retrieval

- $\mathbf{x}_{ij}$ denotes the j-th neuron in the i-th cluster,
- $u(\mathbf{x})_{ij} = 1 \Leftrightarrow \sum_{i'} \sum_{j'} \mathbf{W}_{i'j',ij} \mathbf{x}_{i'j'} \geq s$.
- Storage is performed using Boolean algebra, retrieving is performed using classical linear algebra.

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Memory efficiency (with some approximations)

Approaching $\log(2)$

- Let us choose: $\alpha c = 2\log_2(\ell)$,
- $\eta \sim \frac{nc\log_2(\ell)}{\binom{c}{2}\ell^2} \sim \frac{\alpha n}{\ell^2}$,
- Probability a given connection exists (i.i.d. uniform vectors): $p = 1-(1-\ell^{-2})^n \Rightarrow n \sim -\ell^2\log(1-p)$,
- Probability to accept a random vector: $P_e \approx p^{\binom{c}{2}}$, none of them : $P_e^* \leq P_e \ell^c$,
  - $P_e^* \underset{+\infty}{\leq} \exp\left(\frac{c^2}{2}\left[\log_2(p)+\alpha\right]\right) \to 0$ if $\alpha = -\beta\log_2(p)$, $\beta < 1$.
- Conclusion : $\eta \sim \beta\log_2(1-p)\log_2(p)\log(2)$

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Asymptotic behavior

Storage diversity

Theorem: consider $n = \alpha \log(c)\ell^2$, with $c = \log(\ell)$, then:

- For $\alpha > 2$, a random vector is accepted with probability that goes to 1,
- For $\alpha = 2$, probability is strictly positive,
- For $\alpha < 2$, probability goes to 0.

Stability and error correction

Theorem: Consider $n = \alpha\ell^2/c^2$ vectors. Deactivate ρc initial neurons, then for $\alpha < -\log(1 - \exp(-1/(1-\rho)))$, probability to retrieve the vector goes to 1.

"A comparative study of sparse associative memories," Jour. Stat. Phys.

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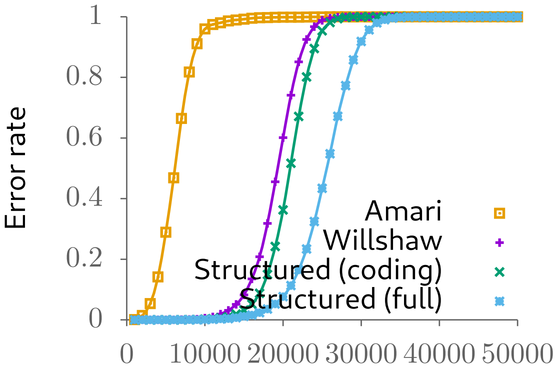
"A comparative study of sparse associative memories," Jour. Stat. Phys.

Performance in search

Amari (Hopfield)
- No structure
- Weights

Willshaw
- No structure
- No weights

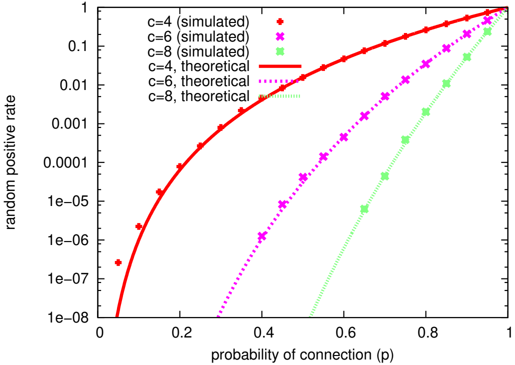
Structured
- Clusters
- No weights



- 2048 neurons total,
- 8 neurons per vector,
- 4 initially activated neurons,
- ($\ell = 256$).

"A comparative study of sparse associative memories," Jour. Stat. Phys.

Performance in indexing



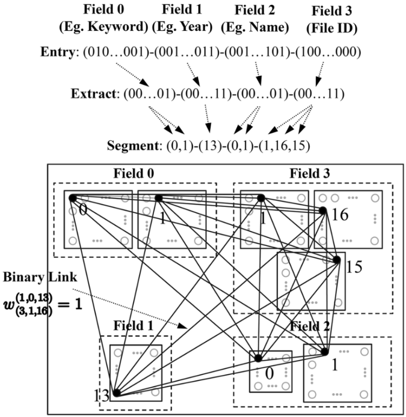
False positive rate for various number of clusters c and $\ell = 512$ neurons per cluster.

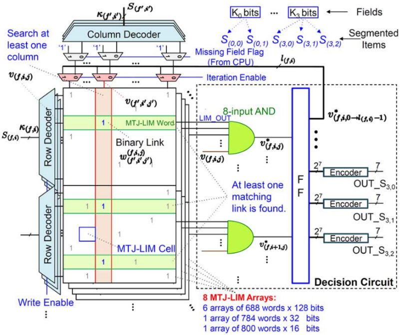
With 1% of error, memory efficiency is 137.1%

Outline

Application to Implementation of Search Engines

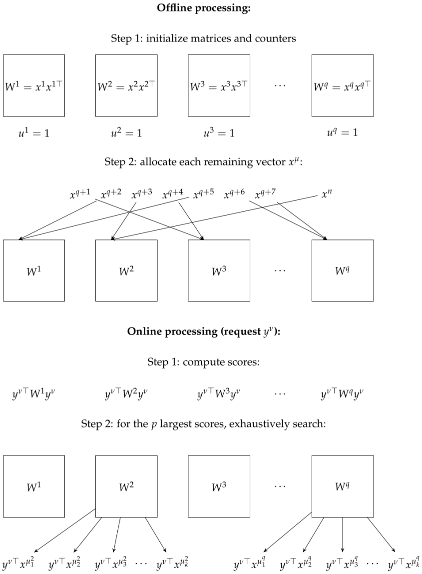
$$v^*_{(f,i,j)} = \left(\bigwedge_{\substack{i'=0 \\ (f',i')\neq(f,i)}}^{\psi-1} \bigvee_{j'=0}^{\gamma-1} w^{(f,i,j)}_{(f',i',j')} v_{(f',i',j')} \right) \bigwedge v_{(f,i,j)}$$





Conclusion: 13.6x memory reduction and 89% energy saving compared to classical CAMs.

"A Nonvolatile Associative Memory-BasedContext-Driven Search Engine Using 90 nmCMOS/MTJ-Hybrid Logic-in-Memory Architecture," IEEE Journal on Emerging and Selected Topics in Circuits and Systems



"Associative Memories to Accelerate Nearest Neighbor Search," Applied Science

Application to Approximate Nearest Neighbor Search

Recalls on SIFT1M dataset.

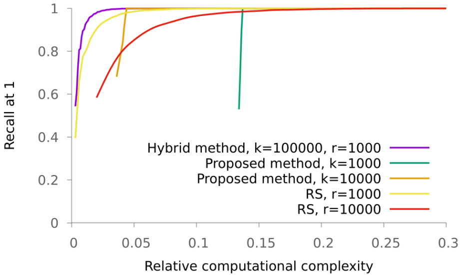


Table 1. Comparison of recall@1 and computation time for one scan (in ms) of the proposed method, kd-trees, K-means trees [1], ANN [16] and LSH [22] on the SIFT1M dataset for various targeted recall performances.

	Scan Time	Recall@1	Scan Time	Recall@1	Scan Time	Recall@1
Random kd-trees [1]	**0.04**	0.6	**0.22**	0.8	3.1	0.95
K-means trees [1]	0.06	0.6	0.25	0.8	2.8	0.99
Proposed method (hybrid)	0.17	0.6	0.25	0.8	**1.1**	0.99
ANN [16]	3.7	0.6	8.2	0.8	24	0.95
LSH [22]	6.4	0.6	11.1	0.8	28	0.98

"Associative Memories to Accelerate Nearest Neighbor Search," Applied Science

DecisiveNets: from DNNs to DAMs

$$\hat{\mathbf{y}}[\ell(i-1):\ell i] = \sigma_t\left(\mathbf{x}[\ell(i-1):\ell i]\right), \forall i, \ \ \texttt{where}$$

$$\sigma_t(\mathbf{z}) = \frac{\texttt{softmax}\left(t \cdot \sigma(\mathbf{z})\right)}{\max\left(\texttt{softmax}\left(t \cdot \sigma(\mathbf{z})\right)\right)}\sigma(\mathbf{z}).$$

	Resnet18 and CIFAR-10		Resnet50 and CIFAR-100	
ℓ	accuracy	multiplications	accuracy	multiplications
1 (baseline)	95.21%	5,070,848	78.50%	1,297,809,408
2	**95.25%**	3,125,248	**79.23%**	861,601,792
4	94.65%	2,152,448	76.58%	643,497,984
8	92.95%	1,666,048	70.46%	534,446,080
16	88.95%	1,422,848	64.36%	479,920,128
32	84.90%	1,301,248	61.07%	452,657,152
64	78.28%	1,240,448	53.05%	439,025,664

Resnet18 and CIFAR10

ℓ	clean data	Gaussian noise	Shot noise	Impulse noise
1	95.21%	46.40%	59.50%	51.75%
2	**95.25%**	47.59%	60.21%	53.45%
4	94.65%	49.98%	61.99%	52.33%
8	92.95%	46.34%	58.07%	53.54%
16	88.95%	50.51%	60.26%	48.95%
32	84.90%	**56.56%**	**64.34%**	**54.78%**
64	78.28%	48.72%	55.60%	40.03%

"DecisiveNets: Training Deep Associative Memories to Solve Complex Machine Learning Problems," in review

Outline

Conclusion

Take-away message

- Structured Clique Networks are very efficient associative memories,
- They can help in many problems,
- They are very efficient for specific hardware.

Interesting directions of research

- Improving explanability, robustness, transferability of knowledge in DNNs,
- DNNs on edge,
- Intricating storing and learning in neural networks,
- Continual learning.

email: vincent.gripon@imt-atlantique.fr